Application of energy balance to compute plasma pinch ratios

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From energy balance the pinch ratio r_p/r_0 (where $r_p =$ equilibrium pinch radius and $r_0 =$ original radius) may be directly computed for the case of a constant current electromagnetic pinch. For the case of a varying current, to solve the energy integral requires the current as a function of the pinch radius. To achieve this an approximate pinch trajectory is first estimated using a snow-plow model. This trajectory by itself gives a zero pinch ratio but it enables the energy integral to be performed. The theoretically predicted values of r_p/r_0 for several cases are found to be in satisfactory agreement with experimental values. This procedure would be useful for designers of pinch devices.

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The plasma pinch has been established as a classical device with several practical applications. However, its theoretical treatment is far from complete. The standard snow-plow model leads to a plasma column of zero radius whilst the Bennett treatment deals with the column as a purely static situation without relationship with the collapse phase. The snow-plow model has been modified to include a kinetic retarding pressure term.[1] Such a model is capable of giving equilibrium pinch radius for the plasma focus in agreement with experimental results. However, the form of the retarding pressure term has not been written in a self-consistent manner and the trajectory is subject to the adjustment of parameters, somewhat arbitrarily.

For a self-consistent estimate of pinch radius ratio r_p/r_0 (where $r_p =$ equilibrium pinch radius, $r_0 =$ original radius) a simple energy balance theory has been suggested.[2] In this communication this theory is reviewed and its application to the computation of radius ratios is demonstrated for the cases of (1) a constant current, constant length pinch, (2) a constant current, varying length pinch, and (3) a constant length, varying current pinch. The results are compared with experimentally measured values.

We consider the following general model for the pinch. On initiation the current I flows axisymmetrically in a thin sheath of radius r_0, as shown in Fig. 1(a). The current acting as a magnetic piston with field B_θ then implodes inwards compressing the plasma sheath which collapses into a column when the front-running shock wave has imploded onto the axis. The magnetic piston continues moving inwards until the temperature of the column becomes high enough to eventually stop the piston motion leading to a quasistatic column at pinch radius r_p.

The magnetic piston exerts a magnetic pressure $P_m = B_\theta^2/2\mu$ where $B_\theta = \mu I/2\pi r$. The work done by the piston in moving from position r_0 to r_p is

$$W = \frac{1}{\rho\pi r_p^2 l_p}\int_{r_p}^{r_0}\frac{\mu^2 I^2}{4\pi^2 r^2(2\mu)}\,2\pi r l\,dr, \tag{1}$$

per unit mass of the final plasma column. Here, in general I and pinch length l are functions of r (or time t) and ρ, r_p, and l_p are the density, radius, and length of the quasiequilibrium pinch.

We write the enthalpy per unit mass of the pinched plasma as

$$h = \frac{P}{\rho}\frac{\gamma}{\gamma-1} = \frac{R_0}{M}T\zeta\frac{\gamma}{\gamma-1}, \tag{2}$$

where $P = (R_0/M)T\zeta\rho$ is the plasma pressure and T, ζ, and γ are the temperature, the departure coefficient, and the specific heat ratio of the pinched plasma, respectively. Also, R_0 is the universal gas constant and M the molecular weight.

Assume that the work done by the magnetic piston is completely converted into the enthalpy of the pinched plasma. Thus, we equate Eq. (1) to Eq. (2) and obtain

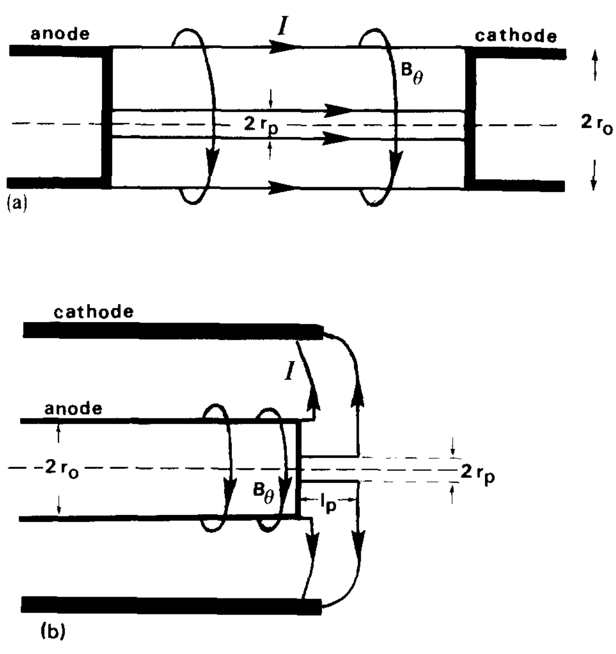

FIG. 1. Comparison of linear Z pinch and plasma focus pinch: (a) linear Z pinch; (b) plasma focus pinch.

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$$T = \frac{\mu}{4\pi^2 \rho r_p^2 l_p} \frac{M}{R_0} \frac{\gamma - 1}{\gamma} \frac{1}{\zeta} \int_{r_p}^{r_0} \frac{I^2 l dr}{r}. \qquad (3)$$

But in its state of quasiequilibrium we may obtain independently another equation for the pinch temperature from pressure balance, $P_m = P$. Thus,

$$T = \frac{\mu I_p^2}{8\pi^2 \rho r_p^2} \frac{M}{R_0} \frac{1}{\zeta}, \qquad (4)$$

where I_p is the current flowing at the time the quasiequilibrium pinch is first established. Combining the independently derived Eqs. (3) and (4) we have

$$I_p^2 = \frac{2(\gamma - 1)}{\gamma l_p} \int_{r_p}^{r_0} I^2 l \frac{dr}{r}. \qquad (5)$$

In general the integral in Eq. (5) which we may call the energy integral may be solved only if I and l are known functions of r.

We shall discuss several specific cases.

Case 1: For this case I = constant, l = constant, and Eq. (5) gives the pinch ratio as

$$\frac{r_p}{r_0} = \exp\left\{ \frac{-\gamma}{2(\gamma - 1)} \right\}. \qquad (6)$$

For $\gamma = 5/3$, Eq. (6) predicts $r_p/r_0 = 0.29$. This case has already been dealt with earlier.[2]

Case 2: This case, of particular interest to a plasma focus, is depicted in Fig. 1(b). As discussed earlier[2] a working empirical relationship between l and r for the plasma focus may be written as

$$l = \frac{l_p}{(r_0 - r_p)} (r_0 - r).$$

For $\gamma = 5/3$, Eq. (5) then gives a value of $r_p/r_0 = 0.14$.

Case 3a current of form $I = I_0 \omega t$: To see the effect of a changing current on the pinch ratio we consider the case $I = I_0 \omega t$. This is the case of a pinch which reaches r_p very early in a sinusoidal capacitor discharge. The effect of this variation in current may be evaluated from Eq. (5) if we have I^2 as a function of r so that the integration in Eq. (5) may be performed. To do this we may use the snow-plow equation:

$$\frac{d}{d\tau} \left\{ (1 - \kappa^2) \frac{d\kappa}{d\tau} \right\} = -\tau^2/\kappa, \qquad (7)$$

where $\kappa = r/r_0$, $\tau = t/t_p$, $t_p = (4\pi^2 \rho_0 r_0^4 / \mu I_0^2 \omega^2)^{1/2}$. This snow-plow model produces a well-known trajectory[3] which yields by itself a nonphysical radius ratio of $r_p/r_0 = 0$ at $\tau = 1.47$. However, this approximate trajectory may be used to give us a relationship between I^2 and κ in the form

$$I^2 = f(\kappa) = 2.180 - 0.797\kappa - 1.835\kappa^2 + 1.081\kappa^3 - 0.955\kappa^4 + 0.326\kappa^5. \qquad (8)$$

Equation (5) may then be written as

$$f(\kappa_p) = 2\left(\frac{\gamma - 1}{\gamma}\right) \int_{\kappa_p}^{1} \frac{f(\kappa)}{\kappa} d\kappa \qquad (9)$$

For $\gamma = 5/3$, the solution of this equation gives a pinch ratio of $r_p/r_0 = 0.17$. We note that in this procedure we have used an approximate trajectory having a nonphysical final pinch ratio of $r_p/r_0 = 0$ to compute a nonzero pinch ratio based on energy balance.

Case 3b $I = I_0 \sin \omega t$: In the same manner we may consider a current of the form $I = I_0 \sin \omega t$ driving the pinch. The snow-plow equation of motion would be

$$\frac{d}{d\tau} \left\{ (1 - \kappa^2) \frac{d\kappa}{d\tau} \right\} = -\alpha^2 \sin^2 \tau/\kappa, \qquad (10)$$

where $\kappa = r/r_0, \tau = t/t_c, \alpha = t_c/t_p, t_c = (L_0 C_0)^{1/2}$, $t_p = \{4\pi^2 \rho_0 r_0^4 / \mu I_0^2\}^{1/2}$.

Solving the trajectory described by this equation of motion, we fit a polynomial of $\sin^2 \tau$ against κ to enable the energy integral to be performed. For $\gamma = 5/3$ and $\alpha = 1$, the pinch ratio was found to be $r_p/r_0 = 0.21$.

Comparison—Case 1: Published results of the Imperial College Mark II Z pinch[4] show that it is a constant length pinch driven by a constant 60-kA current lasting 100 ns. With hydrogen at 0.07 Torr as the test gas, the resulting plasma has $\gamma = 5/3$ and a pinch ratio from streak photographs of $\sim 1/3$ was reported[4] together with a computation[5] predicting a pinch ratio of 0.31. This result has been confirmed in an upgraded device[6] operated with a higher current of 150 kA. Our energy balance theory predicts a pinch ratio of 0.29.

Comparison—Case 2: Published results of the University of Malaya Dense Plasma Focus (UMDPFI) indicates that for a variable length pinch imploding at approximately constant current in deuterium with $\gamma = 5/3$ a pinch ratio of 0.13 was observed from early streak photographs.[2] Recent laser

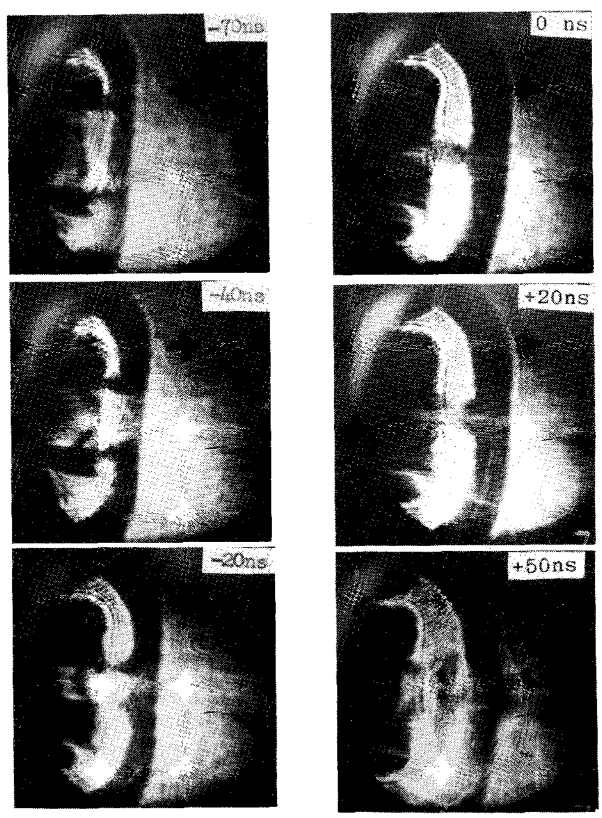

FIG. 2. Side on laser shadowgraph of a 20-kV 60 μF, 4-Torr deuterium plasma focus, showing a pinch ratio $r_p/r_0 \sim 0.13$.

TABLE I. Pinch ratios r_p/r_0 for $\gamma = 5.3$.

		Energy balance theory	Experimental observations
Case 1	constant l, constant I	0.29	~1/3[a]
Case 2	variable l, constant I	0.14	0.13[b]
Case 3a	constant l, $I = I_0\omega t$	0.17	...
Case 3b	constant l, $I = I_0 \sin \omega t$	0.21	0.23[c]

[a] Imperial College.[4,6]
[b] UMDPFI.[2,7]
[c] Bodin *et al.*[8]

shadowgraphs[7] of which a sequence is reproduced in Fig. 2 have confirmed this radius for the UMDPFI. The energy balance theory predicts a pinch ratio of 0.14.

Comparison—Case 3: Many pinch studies made in the early 1960's are of the type in which the current may be approximated as sinusoidal. For example, Bodin *et al.*[8] had trajectories in hydrogen which indicate that for an operating condition close to $\alpha = 1$, a pinch ratio of 0.23 was observed. This compares with a theoretical value of 0.21 from the energy balance theory for a sinusoidal drive current. It should be noted that in Bodin's pinch the current waveform was distorted by the pinch motion so that at the expected peak current time of 1.1 μs the current was depressed below its expected peak value. If this effect were included in the energy balance computation, for example, by coupling the current to the plasma motion through a circuit equation, a pinch ratio greater than 0.21 may be expected from the energy balance.

The results of the above computation based on energy balance are summarized in Table I, with some comparison experimental results.

It would appear that a procedure based on energy balance may be applied to obtain practical estimates on radius ratios for different classes of electromagnetically compressed plasma pinches. In this paper the case of $\gamma = 5/3$ only has been elaborated upon. Equation (6) however predicts lower radius ratios for plasmas having values of γ less than 5/3. This agrees with experimental results in argon[2,6] and nitrogen.[8]

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